Ma2a Practical – Recitation 2

Fall 2024

Exercise 1. For each of the following ODEs, describe if they are undamped, overdamped, underdamped, or critically damped, and sketch the graph of a typical solution.

1. $y'' + y' + y = 0$.
2. $y'' + 2y' + y = 0$.
3. $y'' + 3y = 0$.
4. $y'' + 4y' + 2y = 0$.

Exercise 2. Consider the ODE

$$y'' + 4y' + 3y = \sin(t).$$

Find the general solution, using the reduction of order method and the undetermined coefficients method. Which method is simpler?

Exercise 3. The goal of this problem is to find (all) sequences $(x_n)_{n\in\mathbb{N}}$ of complex numbers satisfying the equation: $x_{n+2} = x_{n+1} + x_n$ for all $n \in \mathbb{N}$.

1. Assume there exists a solution of the form $x_n = a^n$ for some number $a \in \mathbb{C}^*$. Find an equation satisfied by a.
2. Prove that $a = \frac{1+\sqrt{5}}{2}$ or $a = \frac{1-\sqrt{5}}{2}$.
3. Conclude that there exists a 2-dimensional family of solutions.

Solution 1

1. Discriminant $\Delta = 1 - 4 = -3 < 0$, so we get damped oscillations (with angular frequency $\omega = \frac{\sqrt{3}}{2}$). The system is underdamped.

2. Discriminant $\Delta = 4 - 4 = 0$. There are no oscillations, and the system comes back to 0 exponentially. It is critically damped.

3. Discriminant $\Delta = -12 < 0$, so we obtain oscillations. The damping coefficient (the coefficient in front of y') is 0, so the system is undamped. Thus we get steady oscillations at the natural angular frequency of the system $\omega_0 = \sqrt{3}$.

4. Discriminant $\Delta = 16 - 8 = 8 > 0$. The solutions do not oscillate, and since the roots are both negative the solutions decay exponentially. The system is overdamped.

Solution 2

- Method 1: reduction of order. Let $y_1(t) = e^{-t}$, we look for a solution $y = vy_1$. We have

$$y'(t) = v'(t)e^{-t} - v(t)e^{-t},$$
$$y''(t) = v''(t)e^{-t} - 2v'(t)e^{-t} + v(t)e^{-t}.$$

  Plugging back into the equation we have

$$(v'' + 2v')e^{-t} = \sin(t) \Leftrightarrow v'' + 2v' = \sin(t)e^{t}.$$

  Let $w = v'$, we need to solve the ODE $w' + 2w = \sin(t)e^{t}$. We look for an integrating factor $I(t)$, it must satisfy

$$(Iw)' = Iw' + 2Iw \Leftrightarrow (I' - 2I)w = 0.$$

  For example $I(t) = e^{2t}$ works. Then we have $(Iw)'(t) = \sin(t)e^{2t}$. To integrate this, we can write $\sin(t) = \frac{e^{it} - e^{-it}}{2i}$ and integrate the exponentials. After some computations, we obtain

$$I(t)w(t) = \frac{1}{5}(2\sin(t) - \cos(t))e^{2t} + c_1 \Leftrightarrow w(t) = \frac{1}{5}(2\sin(t) - \cos(t)) + c_1 e^{-2t}.$$

  To get v, we need to integrate one more time. We obtain

$$v(t) = \frac{1}{5}(-2\cos(t) - \sin(t)) - \frac{c_1}{2}e^{-2t} + c_2.$$

  Finally we get the general solution

$$y(t) = \frac{1}{5}(-2\cos(t) - \sin(t))e^{-2t} - \frac{c_1}{2}e^{-3t} + c_2 e^{-t}.$$

- <u>Method 2: undetermined coefficients.</u> The discriminant of the characteristic equation is $\Delta = 16 - 12 = 4 > 0$, the roots are $r_- = -3$ and $r_+ = -1$, so the homogeneous solutions are
$$y(t) = Ae^{-3t} + Be^{-t}.$$
We now determine a special solution. Given the form of the inhomogeneous term, we look for a solution of the form $y(t) = a\cos(t) + b\sin(t)$. We then have
$$y'(t) = -a\sin(t) + b\cos(t),$$
$$y''(t) = -a\cos(t) - b\sin(t).$$
Plugging back into the equation we obtain
$$(-a + 4b + 3a)\cos(t) + (-b - 4a + 3b)\sin(t) = \sin(t).$$
Since the functions $(\cos, \sin)$ are linearly independent we must have $4b - 2a = 0$ and $-4a + 2b = 1$. The unique solution is $(a, b) = \frac{1}{5}(-2, -1)$. We obtain the special solution $y(t) = \frac{1}{5}(-2\cos(t) - \sin(t))$, and the general solution is
$$y(t) = \frac{1}{5}(-2\cos(t) - \sin(t)) + Ae^{-3t} + Be^{-t}.$$

<u>Conclusion:</u> the method of undetermined coefficients is faster and leads to easier computations (solving linear systems).